Quadratic Markov modeling for intermittent turbulence

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A Markovian stochastic approach for the relative motion of particle pairs is here considered as a model for intermittency in inertial range turbulence. It is shown that a local acceleration term with quadratic velocity dependence permits a nonclassical scaling of the velocity differences between two points. A Lagrangian model is formulated such that Eulerian flow satisfies incompressibility, experimental second-order, and exact third-order statistics, and is consistent with nonclassical scale similarity. The equation for the Eulerian probability distribution is obtained and discussed. © 1999 American Institute of Physics. [S1070-6631(99)00705-9]

The relative motion of two fluid particles in a turbulent flow represents the Lagrangian counterpart of the classic, Eulerian, local description of turbulence of velocity differences between two points.1,2

Stochastic models for the motion of particle pairs have been proposed recently.3–5 The major effort is devoted to ensure a physical consistency of the modeling, which is purely statistical in the sense that the Navier–Stokes equations are not used. In this Brief Communication; the approach of Refs. 1, 4 is used because it does not require any assumption about the Eulerian probability distribution of velocity differences, which is usually the required unknown. These cited works are based on classical similarity neglecting turbulent intermittency corrections. Even though this may represent a good approximation when building a dispersion model,6 the objective is here to verify the how far a Markovian stochastic model can go to reproduce high-order statistics in turbulence.

We consider locally isotropic homogeneous stationary turbulence of an incompressible flow at an extremely large Reynolds number. We thus neglect the dissipative scales and deal with the classical inertial range.7 We also make use of the Markov hypothesis for the stochastic model. This assumption, which is discussed in length in Refs. 3, 6, cannot be derived from first principles, but it is probably increasingly acceptable at growing values of the Reynolds number, and allows the use of a well established mathematical apparatus.

A stochastic model is given by

\[ \frac{du}{dt} = a(t, r, u)dt + \delta \mu, \quad dr = udtdt; \]  

(1)

where \( u(t) \) is the relative velocity of two fluid particles and \( r(t) \) is their relative position; \( t \) is the time. The function \( a(t, r, u) \) is the local acceleration, or relaxation term. The random forcing \( \delta \mu \), \( \delta \)-correlated in time, is here assumed to be of the Gaussian type with zero mean and covariance matrix \( \delta \mu_i \delta \mu_j = m_{ij}dt \).

Equation (1) corresponds to a forward-Kolmogorov, or Fokker–Planck, equation for the Lagrangian probability distribution \( P_L(u, r, t, r_0) \) of the variables \( u(t) \) and \( r(t) \) given that \( r(0) = r_0 \), and assuming an unbiased Eulerian velocity distribution as an initial condition. Little is actually known about the Lagrangian statistics whereas some information is known for Eulerian ones. Using the relation between Lagrangian and Eulerian probability distributions1,3,4 we obtain an analogous equation for the Eulerian probability distribution \( P_E(u| r) \) of velocity differences (hereafter referred to as just velocity) \( u \) between two points fixed in space,

\[ u_i \frac{\partial P_E}{\partial r_i} + \frac{\partial}{\partial u_i} (a P_E) = \frac{1}{2} \frac{\partial^2}{\partial u_i \partial u_j} (m_{ij} P_E). \]  

(2)

From (2), in inertial range turbulence, we directly get the following model constraints;

\[ \langle a_i \rangle = 0, \]  

(3)

\[ \frac{\epsilon}{3} \delta_{ij} + \langle u_i a_j \rangle + \langle u_j a_i \rangle = -\langle m_{ij} \rangle; \]  

(4)

where angle brackets denote the standard Eulerian ensemble averaging. Equations (3) and (4) have been obtained in Ref. 1 (see also Ref. 4). Constraint (3) is a consequence of incompressibility. Equation (4) contains the exact result from Navier–Stokes on third-order moments (four-fifths law), and \( \epsilon \) is the mean rate of energy dissipation. In summary, a Lagrangian stochastic model satisfying these constraints corresponds to a Eulerian incompressible flow with correct third-order statistics.1,4

Classical similarity assumes that moments of velocity depend on separation with a self-similar scaling \( \langle u^n \rangle \sim (\epsilon r)^{n/3} \). This scaling is exact for third-order moments and fails increasingly with growing \( n \) being rather

\[ \langle u^n \rangle \sim (\epsilon r)^{n/3} f_n(r/l), \]  

(5)

where \( l \) is a typical flow length scale, i.e., the external scale or another significant scale.6

We begin looking for an acceleration term in (1), (2) which is consistent with non-self-similar, intermittent, statistics for the velocity field. Multiplying Eq. (2) by \( u^n \) (\( u \) is the modulus of the velocity), and averaging, we get

\[ \frac{\partial}{\partial r_i} \langle u_i u^n \rangle - n \langle a_i u_i u^{n-2} \rangle = (n/2) \langle m_{ij} u_i u^{n-2} \rangle + (n-2) \langle (n/2) \langle m_{ij} u_i u^{n-4} \rangle. \]  

(6)

The first term in (6) derives from the transport term in Eq. (2); it is model-independent and scales like \( r^{-1} \langle u^{n+1} \rangle \). In order for relaxation \( a \) to be consistent with an intermittent velocity the second term must scale analogously:
\[ \langle au^{n-1} \rangle \sim r^{-1} \langle au^{n+1} \rangle. \]  
(7)

Equations (7) implies that a relaxation with quadratic velocity dependence,
\[ a \sim -r^{-1} u^2, \]  
(8)
allows the possibility of a balance between the relaxation and the transport term independently of the actual scaling of the velocity field, however anomalous it could be.

In Ref. 4 it was assumed that \( a \sim r^{-1} u^2 (r \varepsilon)^{1/3} \), which satisfies (7), (8) only in the special case of classical similarity \( u \sim (r \varepsilon)^{1/3} \). So even though classical similarity is the basic scaling, in order for an intermittent scaling of the velocity field to be possible in the model (1) the acceleration must contain a term quadratic in velocity \( \langle u, \bar{u} \rangle \) zero, i.e.,
\[ \langle u, \bar{u} \rangle = 0; \]  
(9)
because of isotropy.\(^2\)

A first problem occurring when acceleration is proportional to the square of velocity is given by the incompressibility constraint (3). In fact, only the correlation between the longitudinal component \( u_r = u \hat{n}_i, n_i \) being \( r^{-1} r_i \), and the transversal component \( \bar{u}_i = u_r - u \hat{n}_i \) is zero, i.e.,
\[ \langle u_r, \bar{u} \rangle = 0; \]  
(10)
which automatically satisfies the incompressibility constraint (3); the parameters are \( \kappa, \gamma \) which gives anisotropy to the basic linear relaxation, and \( \nu \) representing the relative magnitude of quadratic corrections; it will be seen below that not all these are free parameters. If the random forcing is assumed, in the first instance, to be independent from the velocity, then
\[ m_{ij} = \langle m_{ij} \rangle = \varepsilon (a n, \bar{u} n_i + b (\delta_{ij} - n, n_i)), \]  
(11)
where \( a \) and \( b \) represent the longitudinal and transversal forcing intensity, respectively; it will be shown that their value depends on relaxation parameters. Acceleration (10) is non-linear in \( r \), it is composed of a term linear in velocity, equal to the one introduced in Ref. 4, which implies that classical similarity is the basic scaling behavior and will also balance the random forcing (11) in (6); the additional quadratic term allows the appearance of correction to the similarity.

Relaxation (10) is linearly stable\(^4\) (when \( \kappa > 0 \) and \( \gamma > 0 \)), and the quadratic correction does not lead to instabilities. A qualitative nonlinear analysis can be depicted by writing the acceleration for the longitudinal and modulus of transversal velocities,
\[ du_r / dt = -(\kappa r)^2 (r \varepsilon)^{1/3} (\gamma + 1) u_r + \bar{a}^2 / r, \]  
\[ d\bar{u} / dt = -(\bar{u} / r) (\kappa (r \varepsilon)^{1/3} + (\kappa + 1) u_r). \]  
(12)
These equations show, neglecting the influence of the slower \( r \) evolution, that for large negative longitudinal velocity \( \nu > 0, u_r < -\kappa (r \varepsilon)^{1/3} (1/\kappa + 1) \) a rapid growth can occur in the transversal velocity for a finite time until a large \( \bar{a} \) is able to reduce the magnitude of negative \( u_r \). This local unstable behavior represents particles that rapidly separate as they approach in a manner that is analogous to the presence of a critical point in the flow (whose exact topology, saddle, or focus, depend from the actual relative position and velocity of the approaching particles). An example of the trajectories of particles moving for relaxation only is given in Fig. 1.

The forcing terms \( a \) and \( b \) in (11) are related to the relaxation parameters in view of the further constraint (4). Let us compute the term \( \langle u, \bar{u} \rangle \) in (4), from (10) to yield
\[ \langle u, \bar{u} \rangle = -(\kappa r)^2 (r \varepsilon)^{1/3} (u, u_r) + (\gamma (u, u_r) n_i) \]  
(13)
The two second-order moments can be computed from the general tensor \( \langle u, u_r \rangle \); they depend\(^2\) only on the unknown scalar function \( \langle \bar{u}^2 \rangle \), which can be taken as
\[ \langle \bar{u}^2 \rangle = C (r \varepsilon)^{2/3} (r l)^{\mu}, \]  
(14)
where \( \mu \) is an intermittency exponent, and \( C \) is a coefficient experimentally estimated\(^2\) as \( C = 2 \). The intermittency correction is very small for second-order moments and could be neglected to a good approximation. Third-order moments in (13) can be computed exactly from the four-fifths law,\(^2\)
\[ \langle u, \bar{u}, u_r \rangle = -\frac{3}{15} \varepsilon r (\delta_{ij} - n_i n_j). \]  
(15)
With these specifications expression (13) can evaluated and introduced in constraint (4) to give the forcing coefficients as
\[ a = -\frac{4}{5} + 2 \kappa C (r l)^{\mu} (\gamma + 1), \]  
\[ b = -\frac{4}{5} + \frac{3}{15} \kappa [5 C (r l)^{\mu} - \nu]. \]  
(16)
As pointed out above, the dependence of coefficients \( a \) and \( b \) on separation \( r \) is very weak (\( \mu = 0.03 \)), the marked dependency at small \( r \) is a consequence of neglecting the viscous effect.

The following delicate issue must be noted: in evaluating (13) we have made explicit use of formulas (14) and (15), including their numeric coefficients. There is nothing in the model that guarantees the numeric value of second-order moment which in turn, because of (4), guarantees correct third-order moments. As a consequence the relaxation parameters \( \kappa, \gamma, \) and \( \nu \) are not completely free, every set of parameters must be chosen so that Eq. (14) is verified with the exact coefficients used in (16).
Equation (2) for the Eulerian probability distribution can be rewritten to correspond to the model (10), (11). It is generally a function of three independent variables \( u, \bar{u}, \) and \( r \); furthermore, the classical similarity scaling can be adopted as a normalization. Introducing the dimensionless variables, \( x = u/\langle \varepsilon r \rangle^{1/3}, \ y = \bar{u}/\langle \varepsilon r \rangle^{1/3}, \ \rho = r/l, \) Eq. (2) for the normalized probability \( P(x,y|\rho) = \langle \varepsilon r \rangle P_E(u|r) \) can be rewritten in its dimensionless form,

\[
(y^2 - x^2/3 - \kappa(y + 1)x) (\partial P/\partial x) - (x y + \kappa(y + 3/2)\nu x))P
\]

\[
+ (a/2) (\partial^2 P/\partial x^2) + (b/2) (\partial^2 P/\partial y^2)
\]

(17)

with standard homogeneous boundary conditions in \((x,y)\)-space, normalization constraint, and the additional constraint discussed above about second-order moments \(\langle x^2 \rangle = C\rho^n\) (see Ref. 4). Additional boundary conditions must be specified along \(\rho\), these that introduce a \(\rho\)-dependence on \(P\), correspond to the triggering of an intermittency correction.

Equation (17) for \(\nu = 0\), the absence of intermittency, and the independence from \(\rho\), was solved numerically in Ref. 4. This task is more difficult here because of the additional independent variable \(\rho\). Preliminary numerical results, both Eulerian and Lagrangian, have been obtained neglecting the \(\rho\)-dependence (thus without intermittency). The two dimensional shape of \(P(x,y)\) is analogous to the one obtained in Ref. 4. The one-dimensional probability distribution \(P(x)\) is plotted in Fig. 2 for the case of isotropic forcing, which implies \(y = 1/\lambda - 4\nu/15C\). The Lagrangian probability distribution of one component of particle separation is reported in Fig. 3. Little differences can be noticed on the Eulerian statistics, in the absence of intermittency, and a more compact Lagrangian distribution is found at larger \(\nu\).

Analytical asymptotic solutions of Eq. (17) were presented in Refs. 4, 8; analogous asymptotic approaches can be employed in the present context to determine possible \(\rho\)-dependencies at the tails of the distributions, where intermittency corrections play their role.

The present work represents an approach to the mathematical modeling of intermittency in turbulence. It is shown that a Markov model with relaxation quadratic in velocity supports, in principle, anomalous scaling and a physically consistent geometrical interpretation of trajectories. Along this direction an equation for the Eulerian probability distribution of velocity differences is obtained.

An analysis of this equation may also suggest that a more complex stochastic forcing could be necessary to obtain the observed Lagrangian intermittency properties. In any case the presence of a nonlinear relaxation of this kind will play a role in statistical balances, avoiding that the observed Lagrangian intermittency is just the same of the stochastic forcing.

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